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KAM Theorem for the Nonlinear Schrödinger Equation

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Abstract

We prove the persistence of finite dimensional invariant tori associated with the defocusing nonlinear Schrödinger equation under small Hamiltonian perturbations. The invariant tori are not necessarily small.

1 Introduction

Consider the defocusing nonlinear Schrödinger equation on a circle of unit length,

$$i\partial_t \varphi = -\partial_{xx}\varphi + 2|\varphi|^2\varphi \qquad (t \in \mathbb{R}, x \in S^1). \tag{1}$$

It is a completely integrable Hamiltonian system of infinite dimension with phase space $H^N \equiv H^N(S^1, \mathbb{C})$ $(N \in \mathbb{R}_{\geq 1})$ and Hamiltonian $H_0 \equiv H_0(\varphi, \overline{\varphi})$. Here

$$H^N\left(S^1, \mathbb{C}\right) = \left\{ f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2i\pi kx} \ \mid \ \| f \|_N < \infty \right\},$$

where

$$\| f \|_N^2 := \sum_{k \in \mathbb{Z}} (1 + |k|)^{2N} |\hat{f}(k)|^2$$

and

$$H_0(\varphi, \overline{\varphi}) := \int_0^1 \left(|\varphi_x|^2 + |\varphi|^4 \right) dx.$$

The Poisson structure is given by the regular Poisson bracket

$$\{F, G\} := i \int_{S^1} \left(\frac{\partial F}{\partial \varphi(x)} \frac{\partial G}{\partial \overline{\varphi}(x)} - \frac{\partial F}{\partial \overline{\varphi}(x)} \frac{\partial G}{\partial \varphi(x)} \right) dx,$$

where F, G are functionals on $L^2 \equiv L^2(S^1, \mathbb{C})$ of class C^1. When written as a Hamiltonian system, (1) takes the form

$$\partial_t \varphi = \{H_0, \varphi\} = -i \frac{\partial H_0}{\partial \overline{\varphi}}. \tag{2}$$

Our aim is to prove the existence of quasiperiodic solutions, not necessarily small, of small Hamiltonian perturbations of (2), i.e. of the equation

$$\partial_t \varphi = -i \frac{\partial H}{\partial \overline{\varphi}}, \tag{3}$$

where

$$H(\varphi, \overline{\varphi}) = H_0(\varphi, \overline{\varphi}) + \varepsilon K(\varphi, \overline{\varphi}), \qquad \varepsilon \quad \text{small}. \tag{4}$$

To obtain solutions which are not necessarily close to zero we use a method developed in [11] for the KdV equation: First we prove the existence of global Birkhoff variables $(x_j, y_j)_{j \in \mathbb{Z}}$ (see Section 2). In these new variables, the NLS equation takes the canonical form

$$\begin{cases} \dot{x}_k = -w_k y_k, \\ \dot{y}_k = w_k x_k, \end{cases} \tag{5}$$

where $(\dot{\ })$ denotes the time derivative and $w(I) = (w_k(I))_{k \in \mathbb{Z}}$ is the sequence of frequencies which depend only on the actions $I_j = \left(x_j^2 + y_j^2\right)/2$, $j \in \mathbb{Z}$.

We then verify non-resonance conditions for the frequencies of the unperturbed system (2) reduced by certain symmetries which allow us to apply a refined version [15] of a KAM-theorem of Kuksin [12].

The results of this work have been anounced in [6] and proved in a series of articles [7, 8, 9] and [10].

2 Existence of global Birkhoff variables

It is well known that NLS admits a Lax pair representation

$$\frac{\partial L}{\partial t} = [L, M] := LM - ML, \tag{6}$$

where

$$L = L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \varphi \\ \overline{\varphi} & 0 \end{pmatrix} \tag{7}$$

is the Zakharov–Shabat operator and M is a rather complicated operator given in [4]. As a consequence, the periodic spectrum of $L(\varphi)$, $\mathrm{spec}(\varphi) := \{\lambda \in \mathbb{C}, \exists\ F \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^2)$ with $F \neq 0$, $L(\varphi)F = \lambda F$ and $F(x+2) = F(x)$, $x \in \mathbb{R}\}$, remains invariant under the NLS flow. The periodic spectrum consists of two interlacing sequences $(\lambda_k^+(\varphi))_{k \in \mathbb{Z}}$, $(\lambda_k^-(\varphi))_{k \in \mathbb{Z}}$ of real numbers satisfying $\lambda_k^\pm \sim k\pi$ ($|k|$ large) and $\lambda_k^- \leq \lambda_k^+ < \lambda_{k+1}^-$, $k \in \mathbb{Z}$ (cf [14], [5]). Furthermore, $\mathrm{spec}(\varphi)$ is a complete set of integrals for the NLS equation (cf. [4]). This fact is used to prove the following Theorem (cf. [1] and [7]),

Theorem 1. *For any $N \geq 1$, there exists a bianalytic, bijective symplectomorphism*

$$\Phi : l^2_N\left(\mathbb{Z}, \mathbb{R}^2\right) \to H^N\left(S^1, \mathbb{R}^2\right)$$

such that $(x_j, y_j)_{j\in\mathbb{Z}} = \Phi^{-1}(\varphi)$ *are Birkhoff coordinates for NLS, i.e.* $I_j = \frac{1}{2}\left(x_j^2 + y_j^2\right)$ *are action and* $\theta_j = \operatorname{arctg}\left(\frac{y_j}{x_j}\right)$ *are angle variables.*

Here

$$l^2_N\left(\mathbb{Z}, \mathbb{R}^2\right) := \{(a_j, b_j)_{j\in\mathbb{Z}},\ ||(a_j)_{j\in\mathbb{Z}}||_N + ||(b_j)_{j\in\mathbb{Z}}||_N < +\infty\},$$

where

$$||(a_j)_{j\in\mathbb{Z}}||^2_N = \sum_{j\in\mathbb{Z}} (1+|j|)^{2N}|a_j|^2,$$

and $l^2_N\left(\mathbb{Z}, \mathbb{R}^2\right)$ is endowed with the canonical symplectic structure

$$\Omega((a_j, b_j)_{j\in\mathbb{Z}}, (x_j, y_j)_{j\in\mathbb{Z}}) = 2\sum_{j\in\mathbb{Z}} a_j y_j - b_j x_j.$$

In action angle variables, the Hamiltonian H_0 depends only on the actions, $H_0(\varphi, \overline{\varphi}) = \mathcal{H}(I)$, and NLS is equivalent to the system (5), where $w_k(I) = \frac{\partial \mathcal{H}}{\partial I_k}(I)$.

In particular, given $I \in l^1_{2N}(\mathbb{Z}, \mathbb{R}_+)$ with finite support,

$$\mathcal{T}_I = \Phi\left(\left\{(x_j, y_j)_{j\in\mathbb{Z}},\ x_j^2 + y_j^2 = 2I_j,\ j \in \mathbb{Z}\right\}\right) \tag{8}$$

is an invariant set diffeomorphic to a torus whose dimension is $\sharp\{k \in \mathbb{Z}, I_k \neq 0\}$.

The solution $\phi(x,t) \equiv \varphi_t(x)$ of the initial value problem for NLS, with initial profile $\varphi_0 = \Phi\left(\left(\sqrt{2I_j}e^{i\theta_j}\right)_{j\in\mathbb{Z}}\right)$ in $H^N\left(S^1, \mathbb{C}\right)$, is given by

$$\varphi_t = \Phi\left(\left(\sqrt{2I_j}\ e^{i(\theta_j + t w_j(I))}\right)_{j\in\mathbb{Z}}\right).$$

3 KAM Theorem for NLS

An asymptotic expansion shows that the frequencies have asymptotic resonances,

$$w_{\pm k}(I) \sim 4\pi^2 k^2 \qquad \text{for} \quad k \quad \text{large.}$$

In order to control their effect on perturbed equations we impose symmetry conditions on the perturbation. These conditions (see [8]) allow to consider as phase spaces the subspaces $H^N_\alpha\left(S^1, \mathbb{C}\right)$, $\alpha \in \mathbb{R}$, defined by,

$$H^N_\alpha\left(S^1, \mathbb{C}\right) := \Phi\left(l^2_{N;\alpha}\left(\mathbb{Z}, \mathbb{R}^2\right)\right), \tag{9}$$

where $\left(\sqrt{2I_j}e^{i\theta_j}\right)_{j\in\mathbb{Z}} \in l^2_{N;\alpha}$ iff $\left(\sqrt{2I_j}\ e^{i\theta_j}\right)_{j\in\mathbb{Z}} \in l^2_N$ and satisfies

$$I_{-j} = I_j \qquad \forall\, j \geq 1, \tag{10}$$

and

$$\theta_{-j} \equiv \theta_j + \alpha \ (\mathrm{mod}\ 2\pi) \qquad \forall\, j \geq 0 \quad \text{with} \quad I_j \neq 0. \tag{11}$$

(Notice that for $\alpha \not\equiv 0$ (mod 2π), (11) implies that $I_0(\varphi) = 0$ for all $\varphi \in H^N_\alpha$.)

One verifies that the subspaces $H^N_\alpha\left(S^1, \mathbb{C}\right)$ are invariant under the NLS-flow by showing (cf. [9]) that the symmetries of the NLS Hamiltonian $\mathcal{H}(I)$ imply that $\mathcal{H}(\mathcal{J}(I)) = \mathcal{H}(I)$, where $\mathcal{J}(I)_k = I_{-k}\ \forall\, k \in \mathbb{Z}$. As a consequence, the frequencies $w_j = \frac{\partial \mathcal{H}}{\partial I_j}$ are symmetric at points where $\mathcal{J}(I) = I$.

Moreover, in [8] we provide the following characterization of $H^N_\alpha\left(S^1, \mathbb{C}\right)$,

$$H^N_\alpha\left(S^1, \mathbb{C}\right) = \left\{\varphi \in H^N\left(S^1, \mathbb{C}\right) \mid e^{i\alpha}\check{\varphi} \equiv \varphi\right\},$$

where $\check{\varphi}(x) = \varphi(-x)$. In particular, $H^N_\pi \cap C^\infty$ (resp. $H^N_0 \cap C^\infty$) is the phase-space of $\varphi \in H^N \cap C^\infty$ satisfying generalized Dirichlet (resp. Neumann) conditions, i.e. $\partial_x^{2k}\varphi(0) = \partial_x^{2k}\varphi(1) = 0$ (resp. $\partial_x^{2k+1}\varphi(0) = \partial_x^{2k+1}\varphi(1) = 0$) $\forall k \in \mathbb{Z}$. By a slight abuse of notation, the restriction of Φ to $l^2_{N,\alpha}\left(\mathbb{Z}, \mathbb{R}^2\right)$ is again denoted by Φ.

For $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, a finite subset $A \subseteq \mathbb{Z}_{\geq 0}$ (with $0 \notin A$ if $\alpha \not\equiv 0$) and $I_A \in (\mathbb{R}_{>0})^{|A|}$ we denote by $T^\alpha_{I_A}$ the $|A|$ dimensional torus of the model space $l^2\left(\mathbb{Z}; \mathbb{R}^2\right)$, defined by

$$\begin{aligned} T^\alpha_{I_A} := \Big\{ \left(\sqrt{2J_j}e^{i\theta_j}\right)_{j\in\mathbb{Z}} | J_j = J_{-j} = I_j,\ \forall j \in A; \\ J_j = J_{-j} = 0,\ \forall j \notin A;\ \theta_j = \theta_{-j} + \alpha,\ \forall j \in A \Big\} \end{aligned} \tag{12}$$

and by $\mathcal{T}^\alpha_{I_A}$ the $|A|$ dimensional torus in H^N_α, invariant under NLS,

$$\mathcal{T}^\alpha_{I_A} := \Phi\left(T^\alpha_{I_A}\right). \tag{13}$$

For $\Gamma \subseteq (\mathbb{R}_{>0})^{|A|}$ compact and of positive Lebesgue measure, introduce

$$\mathcal{T}^\alpha_\Gamma := \cup_{I_A \in \Gamma} \mathcal{T}^\alpha_{I_A}. \tag{14}$$

The set $\mathcal{T}^\alpha_\Gamma$ consists of symmetric $2|A|$-gap potentials (if $0 \notin A$) or $(2|A|-1)$-gap potentials (if $0 \in A$), i.e potentials $\varphi \in H^0_\alpha$ with $\lambda^+_j(\varphi) \neq \lambda^-_j(\varphi)$ iff $|j| \in A$ and $\lambda^+_{-j}(\varphi) - \lambda^-_{-j}(\varphi) = \lambda^+_j(\varphi) - \lambda^-_j(\varphi)\ \forall j \geq 1$ (cf. [8]).

We consider Hamiltonian perturbations, $H_\varepsilon = H_0 + \varepsilon K$ on $H^N_\alpha(S^1, \mathbb{C})$ with the following properties:

(P1) K is real analytic on some symmetric neighborhood U_Γ of $\{(\varphi, \overline{\varphi}), \varphi \in \mathcal{T}^\alpha_\Gamma\}$ in $\left(H^N\left(S^1, \mathbb{C}\right)\right)^2$ [1].

(P2) $\frac{\partial K}{\partial \varphi}, \frac{\partial K}{\partial \psi}$ are bounded as functions from U_Γ into $H^N\left(S^1, \mathbb{C}\right)$ and verify the normalization condition

$$\sup\left\{ || \frac{\partial K}{\partial \varphi}(\varphi, \psi) ||_N + || \frac{\partial K}{\partial \psi}(\varphi, \psi) ||_N \mid (\varphi, \psi) \in U_\Gamma \right\} \leq 1.$$

(P3) K satisfies the symmetry condition, $((\varphi, \psi) \in U_\Gamma)$

$$K(\varphi, \psi) = K\left(e^{i\alpha}\check{\varphi}, e^{-i\alpha}\check{\psi}\right).$$

[1] U_Γ is said to be symmetric iff $\left(e^{i\alpha}\check{\varphi}, e^{-i\alpha}\check{\psi}\right) \in U_\Gamma$ for any $(\varphi, \psi) \in U_\Gamma$.

Notice that condition (P3) insures that solutions of $\frac{\partial\varphi}{\partial t} = i\frac{\partial H_\varepsilon}{\partial\varphi}$ for initial data in $H^N_\alpha\left(S^1,\mathbb{C}\right)$ evolve in the same space $H^N_\alpha\left(S^1,\mathbb{C}\right)$.

Our KAM Theorem states that, for ε small enough, many of the NLS-invariant tori $\mathcal{T}^\alpha_{I_A}$ persist under perturbation of the NLS Hamiltonian by εK with K satisfying (P1), (P2), and (P3). Moreover these tori and their linear flows are only slightly perturbed.

Denote by T^n the n-dimensional torus $(\mathbb{R}/\mathbb{Z})^n$.

Theorem 2. *Let $N \geq 1$, A, Γ, α, U_Γ be given as above. Then, for K satisfying (P1), (P2) and (P3), there exists ε_0 so that for any ε with $|\varepsilon| \leq \varepsilon_0$ the following statements hold:*

(i) there exists a Cantor set $\Gamma_\varepsilon \subset \Gamma$ with meas $(\Gamma \setminus \Gamma_\varepsilon) \overset{\varepsilon\to 0}{\to} 0$,

(ii) there exists a Lipschitz family of real analytic torus embeddings

$$\Psi : T^{|A|} \times \Gamma_\varepsilon \to U_\Gamma \cap \left\{(\varphi,\overline{\varphi}) | \varphi \in H^N_\alpha\right\}$$

and

(iii) there exists a Lipschitz map $f : \Gamma_\varepsilon \to \mathbb{R}^{|A|}$

such that for $I_A \in \Gamma_\varepsilon$ and $\theta_A \in T^{|A|}, \Psi(\theta_A + tf(I_A), I_A)$ is a quasiperiodic solution of $\partial_t\varphi = i\frac{\partial H_0}{\partial\overline{\varphi}} + i\varepsilon\frac{\partial K}{\partial\overline{\varphi}}$. Moreover, the deformed invariant tori, $\Psi\left(T^{|A|} \times \{I_A\}\right)$, are linearly stable.

Remarks:

1. Theorem 2 generalizes results due to Kuksin–Pöschel [13] which concern the special case where $\Gamma \subseteq \mathbb{R}^{|A|}_+$ is contained in a sufficiently small neighbourhood of $0 \in \mathbb{R}^{|A|}$ and the phase space consists of elements satisfying generalized Dirichlet boundary conditions. In this situation, action-angle variables are not needed as the Fourier coefficients $(\hat{\varphi}(k))_{k\in\mathbb{Z}}$ are a sufficiently good approximation of the Birkhoff coordinates close to the origin.
2. Similarly, the results of [3] and their generalization in [2], while not directly comparable with our Theorem 2, concern only small perturbations of NLS around $\varphi = 0$ as well.
3. Our results and methods continue the investigation in [11] on the Korteweg-de Vries equation. The purpose of this paper is to document similar features of the NLS equation.

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